

(9) Tensor operators

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① Scalar and vector operators : the definition

- Scalar operator : $U^\dagger(R) S U(R) = S \iff [J_i, S] = 0$
- Vector operator : $U^\dagger(R) \vec{V} U(R) = R \vec{V} \iff [J_i, V_j] = i\hbar \epsilon_{ijk} V_k$

② Tensor operators

- Scalar : rank 0 , Vector : rank 1.
- Rank-n Cartesian Tensor operator: $T_{i_1 j_1 i_2 j_2 \dots}$ $(i_1, j_1, i_2, j_2, \dots = 1, 2, 3)$
n - indices

Rotation : $T_{i_1 j_1 i_2 j_2 \dots} \rightarrow \sum_{i'_1 j'_1 i'_2 j'_2 \dots} \dots R_{i'_1 i_1} R_{j'_1 j_1} R_{i'_2 i_2} R_{j'_2 j_2} \dots T_{i'_1 j'_1 i'_2 j'_2 \dots}$

\rightarrow very complicated! But, it can be simpler in practice.

ex. a "dyadic" tensor : $T_{ij} = U_i V_j$ (rank 2)

\rightarrow can be decomposed into 3 separated notations.

$$U_i V_j = \frac{\vec{U} \cdot \vec{V}}{3} \delta_{ij} + \frac{U_i V_j - U_j V_i}{2} + \left(\frac{U_i V_j + U_j V_i}{2} - \frac{\vec{U} \cdot \vec{V}}{3} \delta_{ij} \right)$$

Scalar op.

anti-symmetrized
 $\sim \epsilon_{ijk} (\vec{U} \times \vec{V})_k$

3x3 symmetric
Traceless tensor.



1 variable

3 var.

5 variables

: Scalars

: vector

: rank-2.



"Reduced"

"irreducible" subspaces

dim = 1 (l = 0)	dim = 3 (l = 1)	dim = 5 (l = 2)
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→ Number of independent components :

reducible $3 \times 3 = 1 + 3 + 5$ irreducible.

→ $(l=1) \otimes (l=1) = (l=0) \oplus (l=1) \oplus (l=2)$

In terms of the irreducible spherical tensors.

③ Vector operator as a spherical tensor.

• Vector operator revisited.

def. $\mathcal{U}^\dagger(R) \vec{V} \mathcal{U}(R) = R \vec{V} \iff [\mathcal{J}_i, V_j] = i\hbar \epsilon_{ijk} V_k$

Rotation in the Hilbert space Rotation in the physical space.

→ $\mathcal{U}(R) = \exp[-\frac{i}{\hbar} \theta (\hat{n} \cdot \vec{J})]$

→ $R = \exp[-i \theta (\hat{n} \cdot \vec{J})]$

✓ $l=1, \langle m | \vec{J} | m' \rangle$

↙ "In Cartesian basis"

$$J_x = \frac{\hbar}{2} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$J_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}$$

See Lecture 18.1

$$J_y = \frac{\hbar}{2} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}$$

$$J_y = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}$$

$(J_i)_{jk} = -i \epsilon_{ijk}$

$$J_z = \hbar \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$J_z = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

DIFFERENT !

"Rotation of a vector"

BUT $\boxed{\vec{J} = U^\dagger \vec{J} U}$

⇒ $\boxed{l=1}$ " in 3D !!!

↑ ↑
Spherical basis Cartesian basis.

U : unitary transformation

→ \vec{J} corresponds to spin-1 angular momentum in the Cartesian basis.

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See, also, $\vec{J}^2 = \begin{pmatrix} 2 & & \\ & 2 & \\ & & 2 \end{pmatrix} \rightarrow \text{eigenvalues} \Rightarrow 2 = j(j+1)$
 $\rightarrow j=1$

(Any classical vector field $A(\vec{x})$,
 like a photon corresponds to spin-1.)

• Spherical basis & the eigenvectors of J_z .

def. $\hat{e}_1 = -\frac{\hat{x} + i\hat{y}}{\sqrt{2}}$, $\hat{e}_0 = \hat{z}$, $\hat{e}_{-1} = \frac{\hat{x} - i\hat{y}}{\sqrt{2}}$

covariant form.



$(l=1) Y_1^{\pm 1} = \mp \sqrt{\frac{3}{4\pi}} \frac{x \pm iy}{r}$, $Y_1^0 = \sqrt{\frac{3}{4\pi}} \frac{z}{r}$

「Spherical Harmonics \in irreducible spherical tensors」
 $[Y_l^m(\vec{r})]$

Let's see if \hat{e}_q indeed belongs to $j=1$.

✓ ① $J_z \hat{e}_q = q \hat{e}_q \parallel q=0, \pm 1 \iff J_z |l, m\rangle = m\hbar |l, m\rangle$

✓ ② $J_{\pm} \hat{e}_q = \sqrt{(1 \mp q)(1 \pm q + 1)} \hat{e}_{q \pm 1} \parallel J_{\pm} = J_x \pm iJ_y$

∴ \hat{e}_q indeed works like $|l=1, m\rangle$ on Y_1^m .

properties of the spherical basis

① $\hat{e}_q = (-1)^q \hat{e}_{-q}^*$

② orthogonality: $\hat{e}_q^* \cdot \hat{e}_q = \delta_{qq'}$

③ Identity $I = \sum_q \hat{e}_q^* \hat{e}_q = \sum_q \hat{e}_q \hat{e}_q^*$
 contravariant, (tensor product.) \hookrightarrow covariant

⊕ Vector $\vec{X} = \sum_q \hat{e}_q^* X_q$, $X_q = \hat{e}_q \cdot \vec{X}$
 ... contravariant.

covariant ... $\vec{X} = \sum_q \hat{e}_q X_q$, $X_q = \hat{e}_q^* \cdot \vec{X}$

→ Rotation: $R \hat{e}_q = \sum_{q', q''} \hat{e}_{q'} \hat{e}_{q''}^* R \hat{e}_{q''} \hat{e}_{q'}^* \cdot \hat{e}_q$
 $= \hat{e}_{q'}^* e^{-i\vec{\theta} \cdot \vec{J}} \hat{e}_{q''} = \delta_{q'q''}$

∴ $R \hat{e}_q = \sum_{q'} \hat{e}_{q'} D_{q'q}^{(1)}(R)$
 $= \langle 1, q' | e^{-i\vec{\theta} \cdot \vec{J}} | 1, q \rangle$
 $= D_{q'q}^{(1)}(R)$

• Irreducible spherical tensor of order 1.

$V_q \equiv T_q^{(1)} = \hat{e}_q \cdot \vec{V}$

rotation: $D^+(R) T_q^{(1)} D(R) = \hat{e}_q \cdot R \vec{V}$
 $= (R^{-1} \hat{e}_q) \cdot \vec{V} = \sum_{q'} \hat{e}_{q'} D_{q'q}^{(1)}(R^{-1}) \cdot \vec{V}$

by setting $R \rightarrow R^{-1}$,

$D(R) T_q^{(1)} D^+(R) = \sum_{q'} T_{q'}^{(1)} D_{q'q}^{(1)}(R)$

⊕ Irreducible spherical tensor operator.

def. $D(R) T_q^{(k)} D^+(R) = \sum_{q'} T_{q'}^{(k)} D_{q'q}^{(k)}(R)$

|| k : rank / order → non-negative INTEGER

(It's the rotation in the physical space!)

def.

commutation relation

• $[J_z, T_q^{(k)}] = \hbar q T_q^{(k)}$

• $[J_{\pm}, T_q^{(k)}] = \hbar \sqrt{(k \mp q)(k \pm q + 1)} T_{q \pm 1}^{(k)}$

generalization

proof with infinitesimal rotations $\mathcal{D}(R) \approx 1 - \frac{i}{\hbar} \theta (\vec{J} \cdot \hat{n})$

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$$\begin{aligned} (1 - \frac{i}{\hbar} \theta (\vec{J} \cdot \hat{n})) T_{\vec{q}}^{(k)} (1 + \frac{i}{\hbar} \theta (\vec{J} \cdot \hat{n})) \\ = \sum_{q'=-k}^k T_{\vec{q}'}^{(k)} \langle k, q' | (1 - \frac{i}{\hbar} \theta (\vec{J} \cdot \hat{n})) | k, q \rangle \end{aligned}$$

$$\Rightarrow [\vec{J} \cdot \hat{n}, T_{\vec{q}}^{(k)}] = \sum_{q'} T_{\vec{q}'}^{(k)} \langle k, q' | \vec{J} \cdot \hat{n} | k, q \rangle //$$

Choose $\hat{n} = \hat{z}$ to get $[J_z, T_{\vec{q}}^{(k)}]$; do similarly for J_{\pm} .

also, one can prove another commutation relation:

$$\Rightarrow \sum_i [J_i, [J_i, T_{\vec{q}}^{(k)}]] = \hbar^2 k(k+1) T_{\vec{q}}^{(k)}$$

⑤ Product of the irreducible spherical tensors

$$T_{\vec{q}}^{(k)} = \sum_{q_1, q_2} \underbrace{\langle k_1, k_2; q_1, q_2 | k, q \rangle}_{= C-G \text{ coeff.}} X_{q_1}^{(k_1)} Z_{q_2}^{(k_2)}$$

It's just like the addition of angular momenta...

$$\text{Ex. } T_0^{(0)} = -\frac{1}{3} \vec{U} \cdot \vec{V}$$

U_q, V_q : rank-1 spherical tensors.

$$T_{\vec{q}}^{(1)} = \frac{1}{\sqrt{2}} (\vec{U} \times \vec{V})_{\vec{q}}$$

$$T_{\pm 2}^{(2)} = U_{\pm 1} V_{\pm 1}$$

$$T_{\pm 1}^{(2)} = \frac{1}{\sqrt{2}} (U_{\pm 1} V_0 + U_0 V_{\pm 1})$$

$$T_0^{(2)} = \frac{1}{\sqrt{6}} (U_{+1} V_{-1} + 2U_0 V_0 + U_{-1} V_{+1})$$

$$\text{Ex. } Y_2^0 = \sqrt{\frac{5}{16\pi}} \frac{3z^2 - r^2}{r^2}$$

$$\text{Since } 3z^2 - r^2 = 2z^2 + 2 \left[-\frac{(x+i\sqrt{2}y)}{\sqrt{2}} \frac{(x-i\sqrt{2}y)}{\sqrt{2}} \right],$$

Y_2^0 is a special case of $T_0^{(2)}$ for $\vec{U} = \vec{V} = \vec{r}$.